Pre-Calculus 30

9.1 Exploring Rational Functions using Transformations

A ***rational function*** is a function that is formed by the ratio of two polynomials. It is written in the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

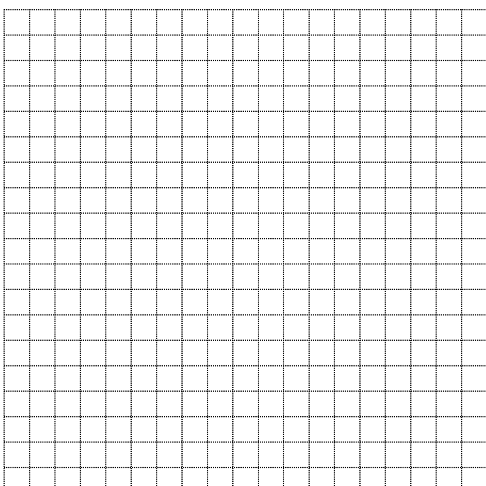
- Why does the definition of a rational function specify $q(x) \neq 0$?

- What will this influence?

What does the graph of a rational function look like? Consider the function $f(x) = \frac{1}{x}$.

Construct a table of values to graph the function.

x													
f(x)													



What happens to the graph as *x* approaches 0?

What happens to the graph as *y* approaches 0?

An ***asymptote*** is a line that the graph of a relation approaches as a limit or boundary.
- The graph of a rational function **<u>never</u>** crosses a vertical asymptote but it may or may not cross a horizontal asymptote.

At what values do vertical asymptotes occur?

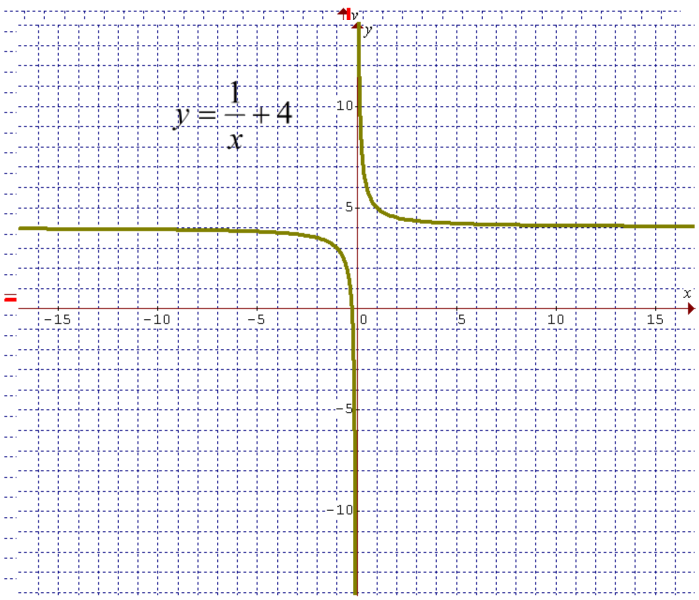
Equation of the vertical asymptote of $f(x) = \frac{1}{x}$:

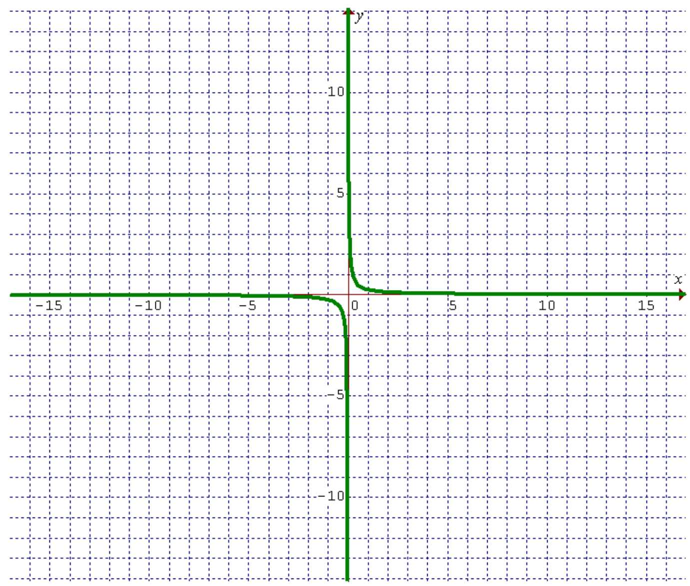
Equation of the horizontal asymptote of $f(x) = \frac{1}{x}$:

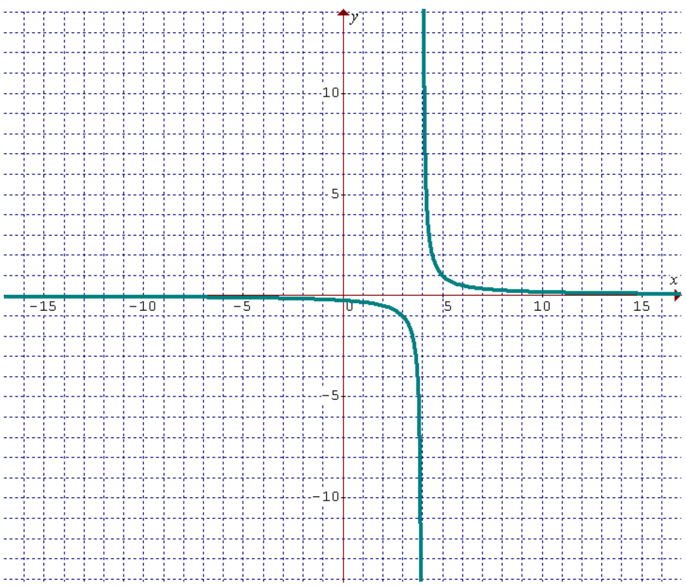
Consider the graphs of the following functions:

$y = \frac{4}{x}$

$y = \frac{1}{}$







How did adding in 4 to the equation of the four functions transform the graph of $f\left(x\right)=\frac{1}{x}$?

Given base function $y=f\left(x\right)$, transformations can be applied using $y=a\ f\left(b\left(x-h\right)\right)+k$.

If we transform the base function $f(x)=\frac{1}{x}$ we get $g(x)=\frac{a}{b(x-h)}+k$

The mapping notation for multiple transformations would be: $\left(x,y\right)\rightarrow($____________, ____________$)$

In general:

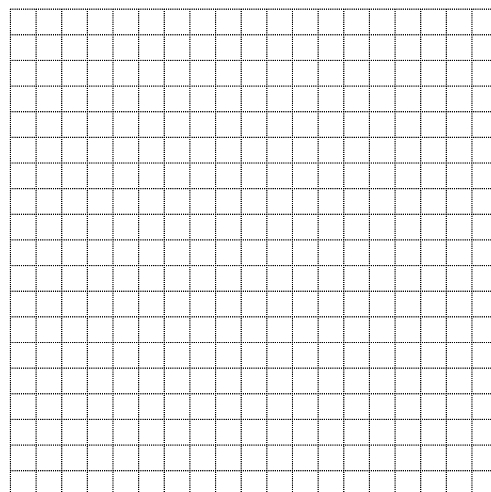
- vertical asymptotes occur at
- horizontal asymptotes occur at

We will apply these transformations to the rational function $f\left(x\right)=\frac{1}{x}$. What are the characteristics of the base function?

- non-permissible value(s)
- behaviour near non-permissible value(s)
- end behaviour
- domain
- range
- equation of vertical asymptote
- equation of horizontal asymptote

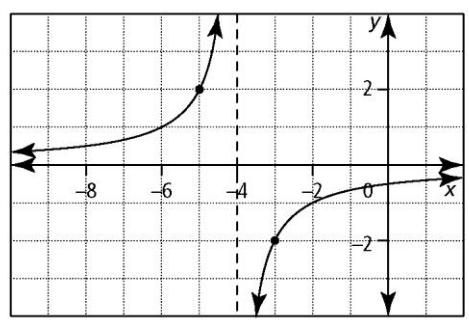
Examples:

1. Sketch the graph of the function $y = \frac{4}{x-2} + 1$ by transforming the graph of $y = \frac{1}{x}$. Will this graph have *x* and/or *y*-intercepts?



2. Write the equation for the function in the form $y = \frac{a}{b(x-h)} + k$

.



<u>Assignment</u>: Page 442 #1, 3, 7abd, 9a

<u>Rewriting Rational Functions</u>

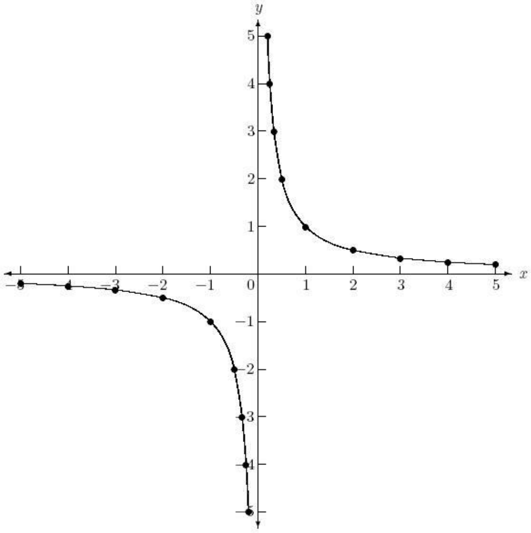
Some rational functions are not in the form $y = \frac{a}{b(x-h)} + k$ so we rewrite it

Note: this only works if there are linear expressions in the numerator and denominator.

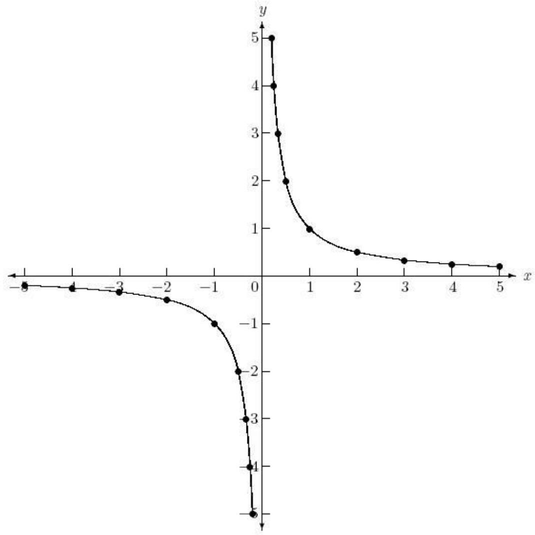
Example:

1. $y = \dfrac{4x-5}{x-2}$

2. $y = \dfrac{2x+2}{x-4}$



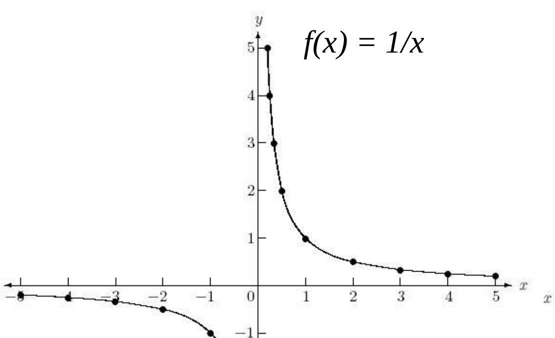
f(x) = 1/x



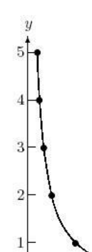
f(x) = 1/x

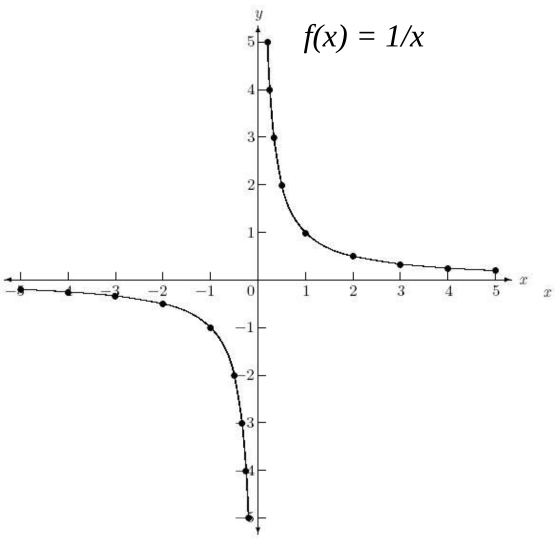
Page 442 #3

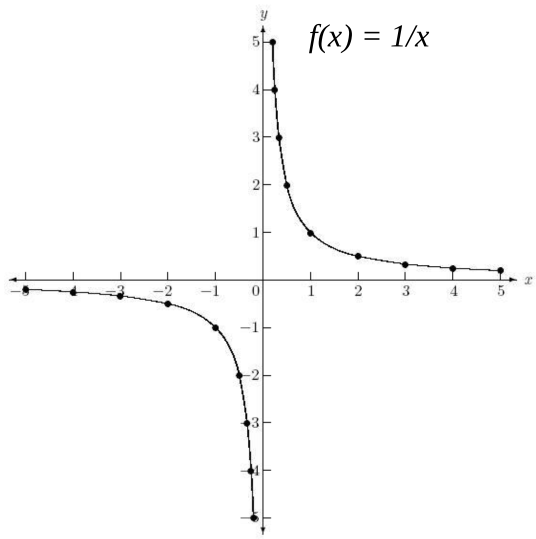
f(x) = 1/x



f(x) = 1/x







Pre-Calculus 30 **9.1 Exploring Rational Functions using Transformations (Day 2)**

We will also use $f\left(x\right)=\frac{1}{x^2}$ as a base function. Construct a table of values to graph this function then identify its characteristics.

x													

f(x)													

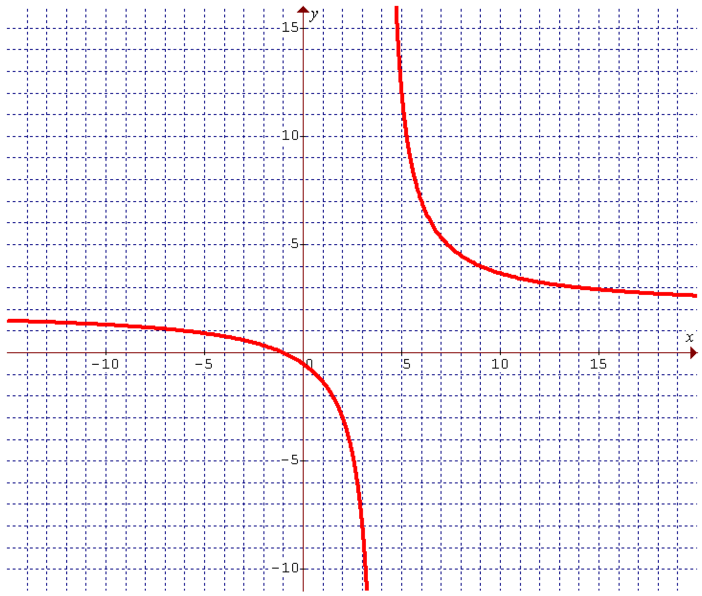
- non-permissible value(s)
- behaviour near non-permissible value(s)
- end behaviour
- domain
- range
- equation of vertical asymptote
- equation of horizontal asymptote

Example:

3. Sketch the graph of the function $y=\dfrac{4}{x^2+6x+9}-2$ by transforming the graph of $y=\dfrac{1}{x^2}$. Will this graph have *x* and/or *y*-intercepts?

Analyzing Functions not written in the Form $y=\dfrac{a}{b(x-h)}+k$ **or** $y=\dfrac{a}{\left(b(x-h)\right)^2}+k$ **:**

Consider the graph of $y = \dfrac{2x+2}{x-4}$.



Identify any asymptotes and intercepts.

How does this graph to relate to the graph of $y = \dfrac{1}{x}$?

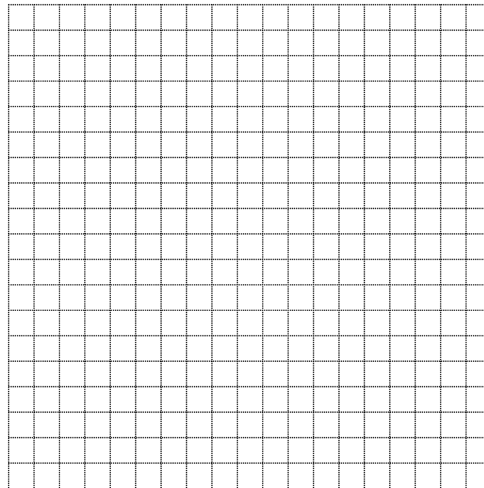
We can use algebra to manipulate the equation of $y = \dfrac{2x+2}{x-4}$ so that it is written in the form $y = \dfrac{a}{b(x-h)} + k$.

This is a time-consuming process and will not work for all types of rational functions. We look at another approach to graphing rational functions in section 9.2 when the function is not given in the form $y = \dfrac{a}{b(x-h)} + k$ or $y = \dfrac{a}{(b(x-h))^2} + k$.

Construct a table of values to graph the function $f\left(x\right)=\frac{1}{x^2}$ then identify its characteristics.

x													
f(x)													

- non-permissible value(s)
- end behaviour (what happens as $|x|$ gets very large?)
- domain
- range
- equation of vertical asymptote
- equation of horizontal asymptote

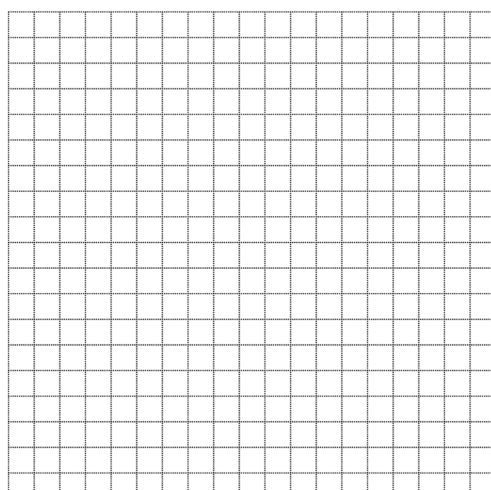


If we transform the base function $f\left(x\right)=\frac{1}{x^2}$ we get $g(x)=\frac{a}{(b(x-h))^2}+k$

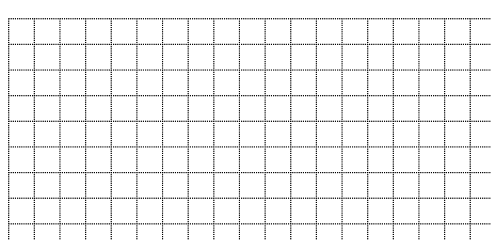
Vertical Asymptotes: ___________________ Horizontal Asymptote: ___________________

If a is negative, the graph is in Quadrants _______ and ________

Example: Sketch the graph of $\frac{-1}{(x-3)^2}$ by transforming the graph of $y=\frac{1}{x^2}$.



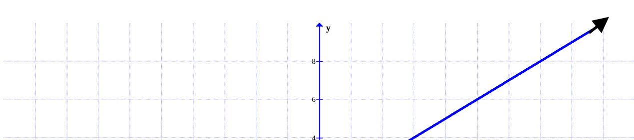
Example: Sketch the graph of the function $y=\frac{4}{x^2+6x+9}-2$ by transforming the graph of $y=\frac{1}{x^2}$. Will this graph have x and/or y-intercepts?



Assignment: Page 442 #2, 6, 8

Pre-Calculus 30 **9.2 Analyzing Rational Functions (Day 1)**

Consider the function $y = \dfrac{x^2 - x - 2}{x - 2}$. What value of *x* is important to consider when analyzing this function?



Is this the graph you anticipated?

How does the behavior of the functions near its non-permissible value differ from the rational functions you looked at previously?

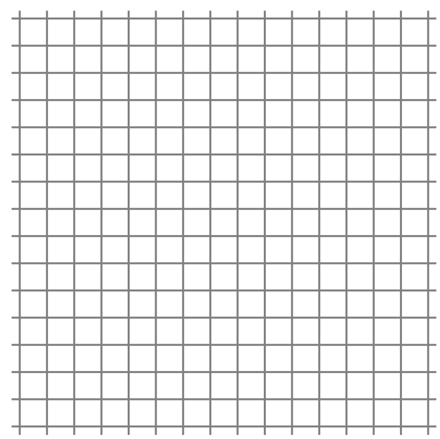
Now, factor the function and reduce.

Does the reduced form show the reason why the graph looks as it does?

Graphs of rational functions can have a variety of shapes and different features – **vertical asymptotes** and **points of discontinuity** (looks like a hole) are possible.

- A **vertical asymptote** occurs where a non-permissible value exists (in the denominator).
- A **point of discontinuity** occurs when the function can be simplified by dividing the numerator and denominator by a common factor that includes a variable. The factor that is eventually reduced is the one that creates the non-permissible value that is the point of discontinuity.

Example: $f(x) = \dfrac{x^2 - 5x + 6}{x - 3}$. Analyze its behavior near its non-permissible value. (Factor, reduce, etc.)



How can you find the **coordinate** of the point of discontinuity?

Find the *x*-intercept

Find the *y*-intercept

Note: The x-intercept is the factor in the numerator!!
See Example 2 on page 448

Assignment: Page 451 #1, 4(predictions only – no graphs)

Pre-Calculus 30 **9.2 Analyzing Rational Functions (Day 2)**

Vertical asymptote(s) or Point(s) of discontinuity

- Occur where the function is undefined; where the denominator = 0.

Horizontal asymptote(s)

- If the numerator and the denominator have the **same degree**, then the equation of the horizontal asymptote line is ***y = ratio of leading coefficients***.

  (ex: $\frac{2x^2+4x}{x^2-4}$*, HA:* $y=\frac{2}{1}=2$ *)*

- If the degree of the numerator is **less than** the degree of the denominator, then the equation of the horizontal asymptote is ***y = 0***.

(These are the only cases we will experience.)

<u>x-intercept</u>

- the factor(s) in the numerator which do not divide out with factors in the denominator.

<u>Ex.</u> Graph $y=\frac{x+4}{x^2+5x+4}$. First factor and state the non-permissible values.

x-intercept(s)

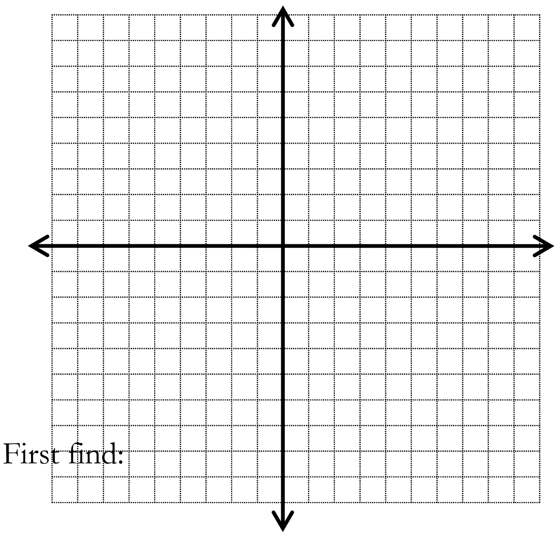
y-intercept(s)

Point(s) of discontinuity:

Vertical asymptote(s):

Horizontal asymptote:

Sign Analysis of **ALL** Factors (where is the graph positive (above the x-axis) and negative (below the x-axis)?)

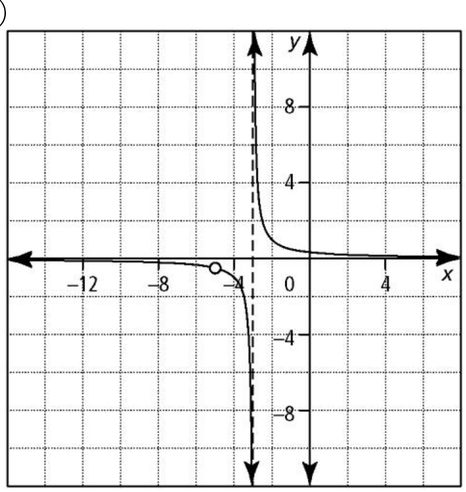


- Place asymptotes and *x*- and *y*-intercepts
- Move graph according to sign analysis
- Use asymptotes as a 'framework'.
- If necessary, find a reference point

<u>Ex.</u> Write the equation for the rational functions shown below. First find:

- Point(s) of Discontinuity
- Vertical Asymptote(s)
- x-intercepts

a)

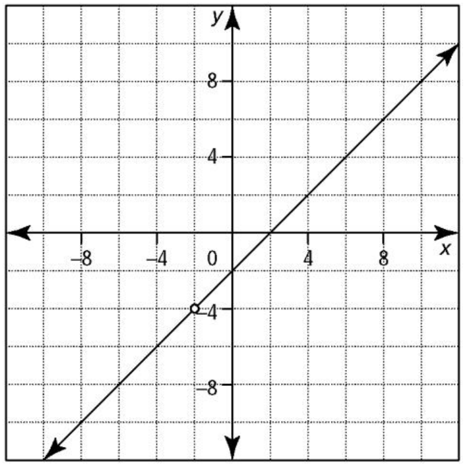


<u>Characteristics</u>

<u>Implications for Factors</u>

Therefore, the equation is…

b)



Characteristics

Implications for Factors

Therefore, the equation is…

Assignment: Page 451 #4abc (graphs), 5 - 8

Pre-Calculus 30 **9.3 Connecting Graphs and Rational Equations**

Rational equations can be solved ***algebraically*** or ***graphically.***

When solving algebraically, watch for ***extraneous roots***.

Example: Solve $\frac{16}{x+6}=4-x$ graphically using technology.

Solving Algebraically:

- Factor numerator and denominator fully. Note any restrictions on variable.
- Eliminate fractions (multiply both sides of equation by the lowest common denominator.)
- Solve for x
- Check for ***extraneous*** roots (use original equation). Does the solution have non-permissible values?
- Write a solution set

Example 1: Solve $\frac{16}{x+6}=4-x$ algebraically.

Example 2: Solve $\frac{x}{2x+5}+2x=\frac{8x+15}{4x+10}$

Assignment: Page 465 #3acd, 5a, 6abd (algebraically only)